
IS EVERYONE READY FOR FUN?
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5.4 Multiple-Angle Identities

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

* $$\cos 2u = \cos^2 u - \sin^2 u$$
$$= 2\cos^2 u - 1$$
$$= 1 - 2\sin^2 u$$

Prove the double-angle identity.

1.) $\cos 2u = 2\cos^2 u - 1$

$= \cos(u+u) = \cos u \cos u - \sin u \sin u$

$= \cos^2 u - \sin^2 u$

$= \cos^2 u - (1 - \cos^2 u)$

$= \cos^2 u + \cos^2 u - 1$

$= 2\cos^2 u - 1$

Prove the identity.

2.) $\sin 3x = \sin \boxed{x}\left(3 - 4\sin^2 \boxed{x}\right)$

$\sin(x + 2x) = \sin x \cos 2x + \cos x \sin 2x$

$= \sin x(1 - 2\sin^2 x) + \cos x(2\sin x \cos x)$

$= \sin x - 2\sin^3 x + 2\sin x \cos^2 x$

$= \sin x - 2\sin^3 x + 2\sin x(1 - \sin^2 x)$

$= \sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

$= \sin x(3 - 4\sin^2 x)$ ✓

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Find the solutions of the equation in the interval $[0, 2\pi)$.

3.) $2\cos x + \sin 2x = 0$

$2\cos x + 2\sin x \cos x = 0$

$2\cos x(1 + \sin x) = 0$

$2\cos x = 0$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$1 + \sin x = 0$

$\sin x = -1$

$x = \frac{3\pi}{2}$

Find the solutions of the equation in the interval $[0, 2\pi)$.

4.) $4\sin x\cos x = 1$

$2\cdot(2\sin x\cos x) = 1$

$2\sin 2x = 1$

$\sin 2x = 1/2$

$\frac{2x}{2} = \frac{\frac{\pi}{6} + 2\pi n}{2}$

$x = \frac{\pi}{12} + \pi n$

$\frac{2x}{2} = \frac{\frac{5\pi}{6} + 2\pi n}{2}$

$x = \frac{5\pi}{12} + \pi n$

$\boxed{[0, 2\pi): \frac{\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12}}$

Find the solutions of the equation in the interval $[0, 2\pi)$.

5.) $\cos x + \cos 3x = 0$

$\cos x + (\cos 2x + x) = 0$

$\cos x + \cos 2x \cos x - \sin 2x \sin x = 0$

$\cos x + (1 - 2\sin^2 x)\cos x - 2\sin x \cos x \sin x = 0$

$\cos x + \cos x \underbrace{- 2\sin^2 x \cos x}\, \underbrace{- 2\sin^2 x \cos x} = 0$

$2\cos x - 4\sin^2 x \cos x = 0$

$2\cos x (1 - 2\sin^2 x) = 0$

$2\cos x = 0$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$1 - 2\sin^2 x = 0$

$\sqrt{\sin^2 x} = \sqrt{\frac{1}{2}}$

$\sin x = \pm \frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Find the solutions of the equation in the interval $[0, 2\pi)$.

6.) $\sin 2x + \sin 4x = 0$

$\sin 2u = 2\sin u \cos u$

$\sin 2x + \sin 2(\overset{u}{2x}) = 0$

$\sin 2x + 2\sin 2x \cos 2x = 0$

$\sin 2x(1 + 2\cos 2x) = 0$

$\sin 2x = 0$

$\frac{2x}{2} = \frac{0 + 2\pi n}{2}$

$x = 0 + \pi n$

$\frac{2x}{2} = \frac{\pi + 2\pi n}{2}$

$x = \frac{\pi}{2} + \pi n$

$0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}$

$\cos 2x = -\frac{1}{2}$

$\frac{2x}{2} = \frac{\frac{2\pi}{3} + 2\pi n}{2}$

$x = \frac{\pi}{3} + \pi n$

$\frac{2x}{2} = \frac{\frac{4\pi}{3} + 2\pi n}{2}$

$x = \frac{2\pi}{3} + \pi n$

$\frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$

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Alternative Methods to Solving #6:

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$$\sin 2x + \sin 4x = 0$$
$$\sin 2x + \sin 2(2x) = 0$$
$$\sin 2x + 2\sin 2x \cos 2x = 0$$
$$2\sin x\cos x + 2(2\sin x\cos x)\,\text{this}\,(1-2\sin^2 x)$$
$$2\sin x\cos x + (4\sin x\cos x)(1-2\sin^2 x)$$
$$2\sin x\cos x + 4\sin x\cos x - 8\sin^3 x\cos x$$
$$6\sin x\cos x - 8\sin^3 x\cos x$$
$$2\sin x\cos x\,(3 - 4\sin^2 x) = 0$$
$$\sin 2x\,(3 - 4\sin^2 x) = 0$$

$$\sin 2x = 0$$
$$\frac{2x}{2} = \frac{0 + 2\pi n}{2}$$
$$x = 0 + \pi n$$
$$x = 0, \pi$$

$$\frac{2x}{2} = \frac{\pi + 2\pi n}{2}$$
$$x = \pi/2 + \pi n$$
$$x = \pi/2,\ 3\pi/2$$

$$\frac{3}{4} = \frac{4\sin^2 x}{4}$$
$$\sqrt{\sin^2 x} = \sqrt{\frac{3}{4}}$$
$$\sin x = \pm\sqrt{3}/2$$
$$x = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}$$

Alternative Methods to Solving #6:

(6) $\sin 2x + \sin 4x = 0$

$2\sin x\cos x + \sin 2(2x) = 0$

$2\sin x\cos x + 2\sin 2x\cos 2x = 0$

$2\sin x\cos x + 2(2\sin x\cos x)(2\cos^2 x - 1)$

$= 2\sin x\cos x + 4\sin x\cos x(2\cos^2 x - 1)$

$= 2\sin x\cos x(1 + 2(2\cos^2 x - 1))$

$= 2\sin x\cos x(1 + 4\cos^2 x - 2)$

$= 2\sin x\cos x(4\cos^2 x - 1)$

$= \sin 2x(4\cos^2 x - 1) = 0$

$\sin 2x = 0$ $\qquad \sqrt{\cos^2 x} = \sqrt{1/4}$

$\cos x = 1/2$

$x = 0, \pi, \pi/2, 3\pi/2, \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Redo #6 using new formulas.

Find the solutions of the equation in the interval $[0,2\pi)$.

7.) $\sin \overset{u}{2x} + \sin \overset{v}{4x} = 0$

$$= 2\sin\left(\frac{2x+4x}{2}\right)\cos\left(\frac{2x-4x}{2}\right) = 0$$

$$= 2\sin(3x)\cos(-x) = 0$$

5.1, 5.2 extension

$\sin 3x = 0$

$x = 0, \pi, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

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Find the solutions of the equation in the interval $[0, 2\pi)$.

8.) $\cos 2x - \cos 6x = 0$

Find the solutions of the equation in the interval $[0, 2\pi)$.

8.) $\cos 2x - \cos 6x = 0$

$= -2\sin(4x)\sin(-2x)$

$= +2\sin(4x)\sin(2x)$

$=$

$\sin 4x = 0$

$x = 0 + \frac{\pi}{2}n$

$x = \frac{\pi}{4} + \frac{\pi}{2}n$

$\sin 2x = 0$

$x = 0 + \pi n$

$x = \frac{\pi}{2} + \pi n$

$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Number 8 without the sum to product formula:

$$\cos 2x - \cos 6x = 0$$

$$\cos 2x - \cos(4x + 2x) = 0$$

$$\cos 2x - (\cos 4x \cos 2x - \sin 4x \sin 2x) = 0$$

$$\cos 2x - (\cos(2x+2x)\cos 2x - \sin 2(2x)\sin 2x) = 0$$

$$\cos 2x - \cos 2x(\cos 2x \cos 2x - \sin 2x \sin 2x) + 2\sin 2x \cos 2x \sin 2x = 0$$

$$\cos 2x - \cos 2x(\cos^2 2x - \sin^2 2x) + 2\sin^2 2x \cos 2x = 0$$

$$\cos 2x - \cos^3 2x + \sin^2(2x)\cos 2x + 2\sin^2(2x)\cos 2x = 0$$

$$\cos 2x(1 - \cos^2 2x + \sin^2(2x) + 2\sin^2(2x)) = 0$$

$$\cos 2x(\underline{1 - \cos^2 2x} + 3\sin^2(2x))$$

$$\cos 2x(\sin^2 2x + 3\sin^2(2x))$$

$$\cos 2x(4\sin^2 2x) = 0$$

$$\cos 2x = 0 \qquad \sin^2(2x) = 0$$

$$\frac{2x}{2} = \frac{\frac{\pi}{2} + 2\pi n}{2} \qquad \sin(2x) = 0$$

$$\frac{2x}{2} = \frac{0 + 2\pi n}{2}$$

$$x = \frac{\pi}{4} + \pi n \qquad x = \pi n$$

$$\frac{2x}{2} = \frac{\frac{3\pi}{2} + 2\pi n}{2} \qquad \frac{2x}{2} = \frac{\pi + 2\pi n}{2}$$

$$x = \frac{3\pi}{4} + \pi n \qquad x = \frac{\pi}{2} + \pi n$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, 0, \pi, \pi/2, 3\pi/2$$

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of $\sin\frac{u}{2}$ and $\cos\frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

Use the half-angle formulas to determine the exact values of the sine, cosine, and tangent of the angle.

$u = \frac{\pi}{4}$

9.) $\frac{\pi}{8}$

$$\sin\frac{u}{2} = \sin\frac{\frac{\pi}{4}}{2} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \sqrt{\frac{2-\sqrt{2}}{2}\cdot\frac{1}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

$$= \frac{1-\cos\pi/4}{\sin\pi/4} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{2}}$$

$$= \frac{2-\sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}-\cancel{2}}{\cancel{2}}$$

$$\sqrt{2}-1$$

Find the solutions of the equation in the interval $[0, 2\pi)$.

10.) $\sin\left(\frac{x}{2}\right) + \cos x - 1 = 0$

$$\sqrt{\frac{1-\cos x}{2}} + \cos x - 1 = 0$$

$$\sqrt{\frac{1-\cos x}{2}}^{\cancel{2}} = (1-\cos x)^2$$

$$\frac{1-\cos x}{2} = 1 - 2\cos x + \cos^2 x$$

$$1 - \cos x = 2 - 4\cos x + 2\cos^2 x$$

$$2\cos^2 x - 3\cos x + 1 = 0$$

$$(2\cos x - 1)(\cos x - 1) = 0$$

$$\cos x = 1/2 \qquad \cos x = 1$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}, 0$$

check for extraneous answers.

Find the solutions of the equation in the interval $[0, 2\pi)$.

11.) $\sin^2 x = 2\left(\sin\frac{x}{2}\right)^2$

$$\sin^2 x = \cancel{2}\left(\cancel{\sqrt{\frac{1-\cos x}{\cancel{2}}}}\right)^{\cancel{2}}$$

$$\sin^2 x = 1 - \cos x$$

$$1 - \cos^2 x = 1 - \cos x$$

$$\cos^2 x - \cos x = 0$$

$$\cos x(\cos x - 1) = 0$$

$\cos x = 0$ $\qquad$ $\cos x = 1$

$x = \pi/2,\ 3\pi/2$ $\qquad$ $x = 0$

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2} \qquad \tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Rewrite as a sum of first powers of the cosines of multiple angles.

12.) $\sin^4 x$

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