8.4 Translation and Rotation of Axes

Write the equation of the conic in standard form.

1.) $16x^2 + 16y^2 - 64x + 32y + 55 = 0$

$16(x^2-4x+4) + 16(y^2+2y+1) = -55+64+16$

$\frac{16(x-2)^2}{16} + \frac{16(y+1)^2}{16} = \frac{25}{16}$

$(x-2)^2 + (y+1)^2 = \frac{25}{16}$

2.) $2x^2 - 3xy + 2y^2 - 2x = 0$

Won't work because of xy-term.

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Rotation

The equation of a conic with axes parallel to one of the coordinate axes has a standard form that can be written in the general form → 1.) $16x^2 + 16y^2 - 64x + 32y + 55 = 0$

$Ax^2 + Cy^2 + Dx + Ey + F = 0.$ Horizontal or vertical axis

Conics whose axes are rotated so that they are not parallel to either the *x*-axis or the *y*-axis have general equations that contain an *xy*-term. 2.) $2x^2 - 3xy + 2y^2 - 2x = 0$

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ Equation in *xy*-plane

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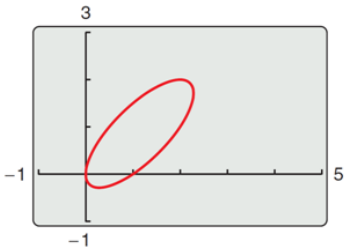
Use the quadratic formula to solve for y and then use a graphing utility to graph the equation.

$2*.)\ 2x^2 - 3xy + 2y^2 - 2x = 0$



$2y^2 - 3xy + (2x^2 - 2x) = 0$

$y = \dfrac{3x \pm \sqrt{9x^2 - 4(2)(2x^2 - 2x)}}{4}$

$y = \dfrac{3x \pm \sqrt{9x^2 - 16x^2 + 16x}}{4}$

$y = \dfrac{3x \pm \sqrt{-7x^2 + 16x}}{4}$

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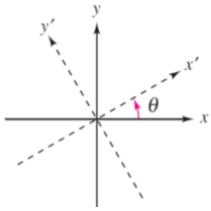
Rotation

To eliminate this *xy*-term, you can use a procedure called **rotation of axes.** The objective is to rotate the *x*- and *y*-axes until they are parallel to the axes of the conic.

The rotated axes are denoted as the *x*′-axis and the *y*′-axis, as shown in Figure 10.32.



After the rotation, the equation of the conic in the new *x*′*y*′-plane will have the form

$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0.$ Equation in *x*′*y*′-plane

Because this equation has no *x*′*y*′-term, you can obtain a standard form by completing the square.

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Rotation of Axes to Eliminate an *xy*-Term

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be rewritten as

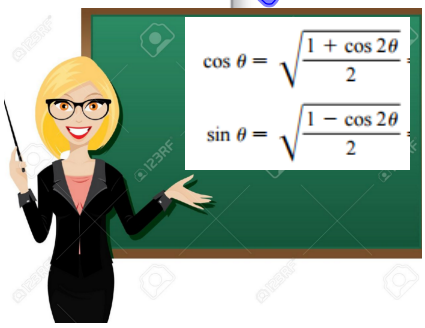
$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

$$\frac{\cot^2 \theta - 1}{2 \cot \theta}$$

I, II



The coefficients of the new equation are obtained by making the substitutions

$$x = x' \cos\theta - y' \sin\theta$$

and

$$y = x' \sin\theta + y' \cos\theta.$$

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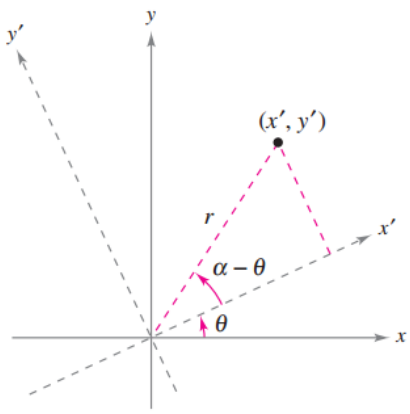

Rotated: $x' = r\cos(\alpha - \theta)$
$y' = r\sin(\alpha - \theta)$

$$\begin{aligned} x' &= r\cos(\alpha - \theta) \\ &= r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ &= r\cos\alpha\cos\theta + r\sin\alpha\sin\theta \\ x' &= x\cos\theta + y\sin\theta \end{aligned}$$

and

$$\begin{aligned} y' &= r\sin(\alpha - \theta) \\ &= r(\sin\alpha\cos\theta - \cos\alpha\sin\theta) \\ &= r\sin\alpha\cos\theta - r\cos\alpha\sin\theta \\ y' &= y\cos\theta - x\sin\theta. \end{aligned}$$

Solving this system for x and y yields

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta.$$

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Rotate the axes to eliminate the xy-term in the equation. Then write the equation in standard form. Sketch the graph showing both axes.

3.) $xy - 2 = 0$

$A = 0$

$B = 1$

$C = 0$

$\cot 2\theta = \frac{A-C}{B} = \frac{0}{1} = 0$

$\frac{2\theta}{2} = \frac{\pi/2}{2}$

$\theta = \pi/4$

$x = x'\cos\theta - y'\sin\theta$

$x = x'\frac{\sqrt{2}}{2} - y'\frac{\sqrt{2}}{2}$

$x = \frac{\sqrt{2}}{2}(x' - y')$

$y = x'\sin\pi/4 + y'\cos\pi/4$

$y = x'\frac{\sqrt{2}}{2} + y'\frac{\sqrt{2}}{2}$

$y = \frac{\sqrt{2}}{2}(x' + y')$

$\frac{\sqrt{2}}{2}(x' - y') \cdot \frac{\sqrt{2}}{2}(x' + y') - 2 = 0$

$\frac{1}{2}(x'^2 - y'^2) - 2 = 0$

$\frac{x'^2}{4} - \frac{y'^2}{4} = \frac{4}{4}$

$\frac{(x')^2}{4} - \frac{(y')^2}{4} = 1$

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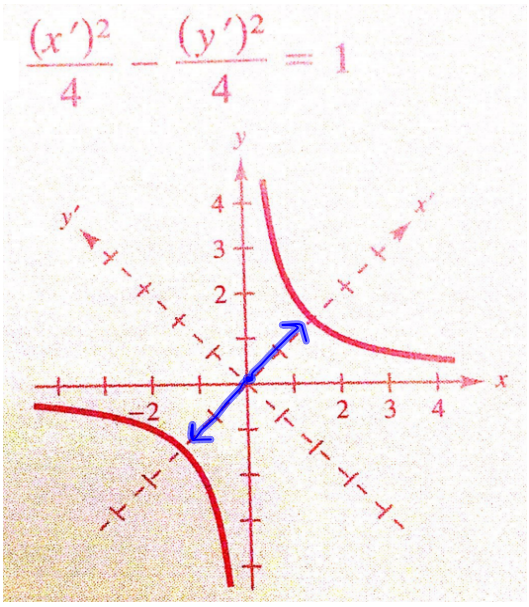

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Rotate the axes to eliminate the xy-term in the equation. Then write the equation in standard form. Sketch the graph showing both axes.

4.) $2\sqrt{3}xy - 2y^2 - \sqrt{3}x - y = 0$

$A = 0$
$B = 2\sqrt{3}$
$C = -2$

$\cot 2\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$ $\tan 2\theta = \sqrt{3}$ $2\theta = \frac{\pi}{3}$ $\theta = \pi/6$

$x = x'\cos\pi/6 - y'\sin\pi/6$ $y = x'\sin\pi/6 + y'\cos\pi/6$

$x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$ $y = \frac{1}{2}(x' + \sqrt{3}y')$

$x = \frac{1}{2}(\sqrt{3}x' - y')$

$2\sqrt{3}\left(\frac{1}{4}\right)(\sqrt{3}x' - y')(x' + \sqrt{3}y') - 2\left(\frac{1}{4}\right)(x' + \sqrt{3}y')(x' + \sqrt{3}y')$

$-\sqrt{3}\left(\frac{2}{2}\right)(\sqrt{3}x' - y') - \left(\frac{2}{2}\right)(x' + \sqrt{3}y')$

$= 2\sqrt{3}(\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2) - 2(x'^2 + 2\sqrt{3}x'y' + 3y'^2)$

$-6x' + 2\sqrt{3}y' - 2x' - 2\sqrt{3}y'$

$= 6x'^2 + 4\sqrt{3}x'y' - 6y'^2 - 2x'^2 - 4\sqrt{3}x'y' - 6y'^2$

$-6x' + 2\sqrt{3}y' - 2x' - 2\sqrt{3}y'$

$4x'^2 - 12y'^2 - 8x' = 0$

$4x'^2 - 8x' - 12y'^2 = 0$

$4(x'^2 - 2x' + 1) - 12y'^2 = 0 + 4$

$\frac{4(x'-1)^2}{4} - \frac{12y'^2}{4} = \frac{4}{4}$

$\frac{(x'-1)^2}{1} - \frac{(y')^2}{1/3} = 1$

$\frac{3y'^2}{1/3}$ $\frac{1}{1}$

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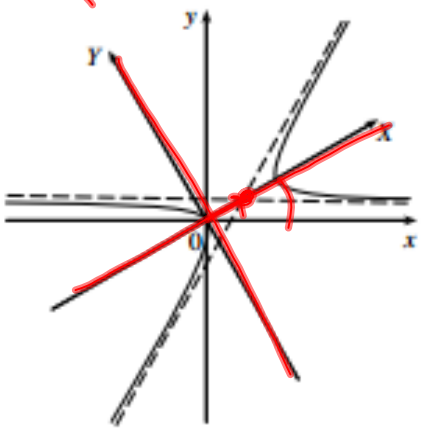
$(X - 1)^2 - 3Y^2 = 1$, hyperbola



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Rotate the axes to eliminate the xy-term in the equation. Then write the equation in standard form. Sketch the graph showing both axes.

5.) $2x^2 - 3xy - 2y^2 + 10 = 0$

$A=2$
$B=-3$
$C=-2$

$\cot 2\theta = \frac{4}{-3}$

$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$

$\frac{-4}{3} = \frac{\cot^2\theta - 1}{2\cot\theta}$

$3\cot^2\theta - 3 = -8\cot\theta$

$3\cot^2\theta + 8\cot\theta - 3 = 0$

$(3\cot\theta - 1)(\cot\theta + 3) = 0$

$\cot\theta = \frac{1}{3}$ $\quad \cot\theta = -3$

SOH
CAH
TOA

$\frac{1}{\sqrt{10}}$ $\frac{3}{\sqrt{10}}$ $\frac{3}{\sqrt{10}}$ $\frac{1}{\sqrt{10}}$

$x = x'\cos\theta - y'\sin\theta$

$x = \frac{1}{\sqrt{10}}(x' - 3y')$

$y = x'\sin\theta + y'\cos\theta$

$y = \frac{1}{\sqrt{10}}(3x' + y')$

$2\left(\frac{1}{10}\right)(x'^2 - 6x'y' + 9y'^2) - 3\left(\frac{1}{10}\right)(3x'^2 - 8x'y' - 3y'^2)$

$-2\left(\frac{1}{10}\right)(9x'^2 + 6x'y' + y'^2) + 10 = 0$

$(x10)$

$2x'^2 - 12x'y' + 18y'^2 - 9x'^2 + 24x'y' + 3y'^2 - 18x'^2 - 12x'y' - 2y'^2 + 100 = 0$

$\frac{-25x'^2}{-100} + \frac{25y'^2}{-100} = \frac{-100}{-100}$

$\frac{(x')^2}{4} - \frac{(y')^2}{4} = 1$

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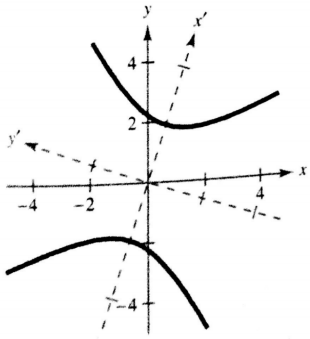
$$\frac{(x')^2}{4} - \frac{(y')^2}{4} = 1$$



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Rotation Invariants

The rotation of the coordinate axes through an angle θ that transforms the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

has the following rotation invariants.

1. $F = F'$
2. $A + C = A' + C'$
3. $B^2 - 4AC = (B')^2 - 4A'C'$

Apr 17-10:32 AM

Classification of Conics by the Discriminant

The graph of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is, except in degenerate cases, determined by its discriminant as follows.

1. *Ellipse or circle:* $B^2 - 4AC < 0$
2. *Parabola:* $B^2 - 4AC = 0$
3. *Hyperbola:* $B^2 - 4AC > 0$

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Use the discriminant to classify the graph.

6.) $12x^2 - 6xy + 7y^2 - 45 = 0$

7.) $x^2 - 6xy - 5y^2 + 4x - 22 = 0$

8.) $36x^2 - 60xy + 25y^2 + 9y = 0$

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Apr 17-11:07 AM